One-Dimensional N-body Problem with δ-Interaction

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(Z. Naturforsch. **25 a**, 127—134 [1970]; received 16 July 1969)

We have studied the one dimensional N-body problem with delta-function interaction and width $2C$.

To determine the ground-state energy and the excitation spectrum we arrive at integral equations similar to the equation of Hulthen.

The solution of the integral equation is achieved by the moments method and the results are compared with those found in the literature.

The problem of N-mutually interacting particles of different masses in one dimension has been studied by various authors. A comprehensive literature may be found in the recently published by LIEB-MATTIS [1], "Mathematical physics in one dimension".

For many cases of the interacting potential the above problem may be solved exactly. These cases have been studied and are mentioned in a paper by MCGUIRE [2].

The system of N-fermions spin 1/2 of the same mass and $\delta(x_i - x_j)$ interaction potential, distributed over a circle of radius L, has been recently studied by GAUDIN [3].

Furthermore, Gaudin gives the equations for the calculation of the energy state of the system for all values of total spin S.

For the special case of attractive potential, in which the fermions act in pairs, the energy states of those pairs coincide with the energy states of a boson gas with the same interaction $\delta(x_i - x_j)$ of width $2C$. Both the density of energy states and the ground-state energy E for spin zero are given by the same expressions.

For the case of boson gas in a repulsive potential the above quantities have been calculated by LIEB-LINIGER [4] and are given by the following expression,

$$\varrho = \frac{1}{\pi} \int_{-Q}^{Q} f(q)\, dq\,, \qquad (1)$$

$$E = -\frac{1}{4} C^2 + \frac{1}{\pi \varrho} \int_{-Q}^{Q} q^2 f(q)\, dq$$

where the function $f(q)$ satisfies the integral equation,

$$\frac{1}{2} f(q) = 1 + \frac{C}{2\pi} \int_{-Q}^{Q} \frac{f(q')\, dq'}{(q-q')^2 + C^2}\,. \qquad (2)$$

Integral equations similar to the above have been found by HULTHEN [5] and have been applied to problems of Ferromagnetism and Antiferromagnetism [6]. In the present paper we will study the solution of Eq. (2) which we write in the usual way as:

$$2\pi g(y) = 1 + 2\lambda \int_{-1}^{1} \frac{g(x)\, dx}{\lambda^2 + (x-y)^2}\,. \qquad (3)$$

From this by changing the variables we can derive (2) and vice versa [4].

For the solution of the integral equation (3) we shall use the moments method [7] and then apply the results to determine the ground-state energy and the excitation energies of different spins.

1. Investigation of the Integral Equation

As is known the integral equation (3) is a special case of the following integral equation

$$r(y) + \int_{-1}^{1} K(y-x)\, g(x)\, dx = \sigma g(y) \qquad (4)$$

with integral kernel

$$K(y-x) = 2\lambda / [\lambda^2 + (x-y)^2]\,. \qquad (5)$$

The general theory for the Eq. (4) is given in the literature of integral equations [8]. The existence

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[1] E. LIEB and D. MATTIS, Mathematical Physics in One Dimension, Academic Press, New York—London 1966.

[2] J. MCGUIRE, J. Math. Phys. **5**, 622 [1964].

[3] M. GAUDIN, Phys. Letters **24 A**, 55 [1967].

[4] E. LIEB and W. LINIGER, Phys. Rev. **130**, 1605 [1963].

[5] L. HULTHEN, Arkiv Math. Astron. Fys. **26 A**, No 11 [1938].

[6] R. GRIFFITHS, Phys. Rev. **133**, 768 A [1964].

[7] V. VOROBYEV, Method of Moments in Applied Mathematics, Gordon & Breach Science Publ. Inc., New York 1965.

[8] R. COURANT and D. HILBERT, Methoden der Mathematischen Physik, Springer-Verlag, Berlin 1931.

of a solution as well as the analyticity of the solution with respect to the parameter λ has been studied by LIEB-LINIGER [4].

Taking the above into consideration we will attempt to find the solution (3) by the use of the moments method which has recently been developed and applied by CHI-YU-HU [9] for the calculation of ground-state energy of different physical problems. In addition it has been shown that the zero approximation of this method corresponds to the third order approximation of a normal perturbation method.

Equation (3) can be written as:

$$2\pi g(y) = 1 + i \int_{-1}^{1} g(x) \left\{ \frac{1}{x-y+i\lambda} - \frac{1}{x-y-i\lambda} \right\} dx . \tag{6}$$

We define the moments as

$$I_m = \int_{-1}^{1} x^m g(x) \, dx \tag{7}$$

and expanding the functions $1/(x-y+i\lambda)$, $1/(x-y-i\lambda)$ by MacLaurin's series (it is understood that they converge), we obtain

$$\frac{1}{x-y+i\lambda} - \frac{1}{x-y-i\lambda} \tag{8}$$
$$= \sum_{n=0}^{\infty} x^n \left\{ \frac{1}{(y+i\lambda)^{n+1}} - \frac{1}{(y-i\lambda)^{n+1}} \right\} .$$

By substitution of (8) and (7) into (6) we obtain the following solution

$$2\pi g(y) = 1 + i \sum_{n=0}^{\infty} I_n \left\{ \frac{1}{(y+i\lambda)^{n+1}} - \frac{1}{(y-i\lambda)^{n+1}} \right\} . \tag{9}$$

Finally by substituting the above solution into (7), we obtain the following system from which the coefficients I_m can be calculated

$$2\pi I_m = \frac{1+(-1)^m}{m+1} + 2 \sum_{n=0}^{\infty} I_n B_{n+1,m}(\lambda) \tag{10}$$

with $B_{n,m}(\lambda)$ representing the integral

$$B_{n,m}(\lambda) = \frac{i}{2} \int_{-1}^{1} x^m \left\{ \frac{1}{(x+i\lambda)^n} - \frac{1}{(x-i\lambda)^n} \right\} dx . \tag{11}$$

It can easily be proved that the coefficients $B_{n,m}(\lambda)$ satisfy the following conditions:

$$B_{2n,2m}(\lambda) = 0 , \; B_{2n+1,2m}(\lambda) \neq 0 ,$$
$$B_{2n+1,2m+1}(\lambda) = 0 . \; B_{2n,2m+1}(\lambda) \neq 0 . \tag{12}$$

Owing to the above relations the system (10) breaks-up into two new systems of even and odd indices. Thus:

$$\pi I_{2m} = \frac{1}{2m+1} + \sum_{n=0}^{\infty} I_{2n} B_{2n+1,2m}(\lambda) , \tag{13}$$

$$\pi I_{2m+1} = \sum_{n=0}^{\infty} I_{2n+1} B_{2n+2,2m+1}(\lambda) . \tag{14}$$

The system (14) is linear and homogeneous, the condition for its solution being the vanishing of the determinant. From this condition we can calculate the eigenvalues of the parameter λ and then from the system (13) we can fied the coefficient I_{2n}.

In many physical problems the values of the parameter λ is defined from the solution of one more equation, e. g. in the case of Lieb-Liniger λ satisfies the following equations:

$$\lambda = \gamma \int_{-1}^{1} g(x) \, dx = \gamma I_0(\lambda) . \tag{15}$$

Since it is possible to have eigenvalues which are solutions of (15) and for which the determinant of system (14) is non-vanishing, it follows that (14) has the zero soluiton, i. e. the odd coefficients are zero.

The above result may be expected on account of the existing symmetry of the problem.

The functions $I_0(\lambda)$, $I_2(\lambda)$, which are required for the calculation of the eigenvalues of λ as well as for the calculation of other physical quantities, are determined by solving the following system:

$$\begin{array}{llllll} \pi I_0 = & 1 & + B_{1,0} I_0 & + B_{3,0} I_2 & + \ldots + B_{2n+1,0} I_{2n} & + \ldots \\ \pi I_2 = & \frac{1}{3} & + B_{1,2} I_0 & + B_{3,2} I_2 & + \ldots + B_{2n+1,2} I_{2n} & + \ldots \\ \ldots & & & & & \\ \pi I_{2m} = & \frac{1}{2m+1} & + B_{1,2m} I_0 & + B_{3,2m} I_2 & + \ldots + B_{2n+1,2m} I_{2n} & + \ldots . \end{array} \tag{16}$$

Before we attempt a solution of the system we must examine the coefficients $B_{2n+1,2m}(\lambda)$ as functions of the parameter λ and calculate then.

[9] CHI-YU-HU, Phys. Rev. **152**, 1116 [1966].

2. Calculation of the Coefficients $B_{2n+1,2m}(\lambda)$

The calculation of the coefficients $B_{2n+1,2m}(\lambda)$ is elementary since by their definition we have,

$$B_{2n+1,2m}(\lambda) = \frac{i}{2}\int_{-1}^{1} x^{2m} \left\{\frac{1}{(x+i\lambda)^{2n+1}} - \frac{1}{(x-i\lambda)^{2n+1}}\right\} dx \tag{17}$$

and for $n=0$:

$$B_{1,2m}(\lambda) = \frac{i}{2}\int_{-1}^{1} x^{2m} \left\{\frac{1}{(x+i\lambda)} - \frac{1}{(x-i\lambda)}\right\} dx$$

or

$$B_{1,2m}(\lambda) = 2\lambda\left\{\sum_{l=0}^{m-1} \frac{(-\lambda^2)^l}{2m-2l-1} + \frac{(-\lambda^2)^m}{\lambda}\operatorname{arctg}\frac{1}{\lambda}\right\}. \tag{18}$$

Differentiating $B_{1,2m}(\lambda)$ $(2n)$ times with respect to λ we obtain:

$$B_{2n+1,2m}(\lambda) = \frac{(-1)^n}{(2n)!}\frac{d^{2n} B_{1,2m}(\lambda)}{d\lambda^{2n}}. \tag{19}$$

The above formula is very useful because it allows the computation of all the coefficients, by differentiation with respect to λ, for any positive values of n, m. We write below the first coefficients:

$$B_{1,0} = 2\operatorname{arctg}(1/\lambda),\ B_{3,0} = -2\lambda/(1+\lambda^2)^2, \ldots,\ B_{2n+1,0} = -\frac{i}{2n}\left\{\frac{1}{(1+i\lambda)^{2n}} - \frac{1}{(1-i\lambda)^{2n}}\right\},$$

$$B_{1,2} = 2\lambda\left\{1-\lambda\operatorname{arctg}\frac{1}{\lambda}\right\},\quad B_{3,2} = 2\left\{\operatorname{arctg}\frac{1}{\lambda} - \frac{1}{\lambda} + \frac{1}{\lambda(1+\lambda^2)^2}\right\}, \ldots . \tag{20}$$

A characteristic property of the coefficients $B_{2n+1,2m}(\lambda)$ is their asymptotic behaviour for large λ, since they tend to zero.

The coefficient $B_{1,2m}(\lambda)$ for $\lambda>1$ is easily computed as a series of inverse powers of λ, i. e.

$$B_{1,2m}(\lambda) = \frac{2}{\lambda}\sum_{l=0}^{\infty}\frac{(-1)^l}{2m+2l+1}\frac{1}{\lambda^{2l}} \tag{21}$$

and from formula (19) we have

$$B_{2n+1,2m}(\lambda) = \frac{2(-1)^n}{(2n)!}\sum_{l=0}^{\infty}\frac{(-1)^l}{2m+2l+1}\frac{d^{2n}}{d\lambda^{2n}}\left(\frac{1}{\lambda^{2l+1}}\right). \tag{22}$$

We can now solve system (16) by approximations and apply the results to determine the ground-state energy of an N-particle interacting system.

3. Ground-State Energy of the System

To compute the ground-state of a system of N-particles interacting in potential of the form $2C\,\delta(x_i-x_j)$ we need, according to LIEB-LINIGER [4], the zero order moment

$$I_0 = \int_{-1}^{1} g(x)\,dx \tag{23}$$

and the second order moment:

$$I_2 = \int_{-1}^{1} x^2 g(x)\,dx. \tag{24}$$

The values of the parameter λ are determined from the eigenvalue equation

$$I_0(\lambda) = \lambda \tag{25}$$

and the ground-state energy is given by

$$E_0 = N\varrho^2 e(\gamma) \tag{26}$$

where

$$e(\gamma) = (\gamma^3/\lambda^3)\int_{-1}^{1} x^2 g(x)\,dx = (\gamma^3/\lambda^3)\, I_2(\lambda), \tag{27}$$

$\gamma = C/\varrho$ and ϱ represents the ratio N/L (N is the number of particles and L the radius of the circle).

For $\lambda>1$ we can consider the zero approximation as a solution namely:

$$I_0(\lambda) = \frac{1}{\pi - B_{1,0}} = \frac{1}{\pi - 2\operatorname{arctg}(1/\lambda)},$$

$$I_2(\lambda) = \frac{1}{3\pi} + \frac{B_{1,2}}{\pi(\pi - B_{1,0})}, \tag{28}$$

$$\cdots\cdots\cdots$$

$$I_{2n}(\lambda) = \frac{1}{\pi(2n+1)} + \frac{B_{1,2m}}{\pi(\pi - B_{1,0})}.$$

Because of (28) Eqs. (25) and (26) take the form:

$$\gamma = \lambda \left(\pi - 2 \operatorname{arctg} \frac{1}{\lambda} \right), \qquad (29)$$

$$e(\gamma) = \frac{\gamma^3}{\pi \lambda^3} \left\{ \frac{1}{3} + \frac{2 \lambda (1 - \lambda \operatorname{arctg}(1/\lambda))}{\pi - 2 \operatorname{arctg}(1/\lambda)} \right\} \cdot \qquad (30)$$

For very large λ, the aboe equation becomes:

$$\lambda = \frac{1}{\pi} (\gamma + 2), \qquad (31)$$

$$e(\gamma) = \tfrac{1}{3} \pi^2 (\gamma / (\gamma + 2))^2 . \qquad (32)$$

These are exactly the results of LIEB-LINIGER [4]. We obtain a better expression for $e(\gamma)$ by solving system (16) up to $B_{3,2m}$, i. e.

$$I_0(\lambda) = \frac{\pi - B_{3,2} + \frac{1}{3} B_{3,0}}{(\pi - B_{1,0})(\pi - B_{3,2}) - B_{1,2} B_{3,0}} = \lambda / \gamma , \qquad (33)$$

$$I_2(\lambda) = \frac{B_{1,2} + \frac{1}{3}(\pi - B_{1,0})}{(\pi - B_{1,0})(\pi - B_{3,2}) - B_{1,2} B_{3,0}} = (\lambda^3/\gamma^3)\, e(\gamma) . \qquad (34)$$

Substituting $B_{1,0}$, $B_{1,2}$, $B_{3,0}$ and $B_{3,2}$ from (20) we obtain

$$\left(\pi - 2 \operatorname{arctg} \frac{1}{\lambda} \right) \left\{ \pi - 2 \operatorname{arctg} \frac{1}{\lambda} + \frac{2}{\lambda} \left(1 - \frac{1}{(1+\lambda^2)^2} \right) \right\} + \frac{4 \lambda^2}{(1+\lambda^2)^2} \left(1 - \lambda \operatorname{arctg} \frac{1}{\lambda} \right)$$
$$= \frac{\gamma}{\lambda} \left\{ \pi - 2 \operatorname{arctg} \frac{1}{\lambda} + \frac{2}{\lambda} \left(1 - \frac{1}{(1+\lambda^2)^2} \right) - \frac{2 \lambda}{3 (1+\lambda^2)^2} \right\}, \qquad (35)$$

$$e(\gamma) = \frac{\gamma^2}{\lambda^2} \cdot \frac{2 \lambda [1 - \lambda \operatorname{arctg}(1/\lambda)] + \frac{1}{3} [\pi - 2 \operatorname{arctg}(1/\lambda)]}{\pi - 2 \operatorname{arctg}(1/\lambda) + (2/\lambda)(1 - 1/(1+\lambda^2)^2) - 2 \lambda / 3 (1+\lambda^2)^2}, \qquad (36)$$

these are approximately valid for all $\lambda > 0$. The case $\lambda \gg 1$ leads again to (31), (32). The system (16) for small values of λ by the substitution

$$B_{2n+1, 2m}(\lambda) = 2 \frac{2n+1}{2m - 2n - 1} \lambda \quad \text{for} \quad m \neq n, \qquad B_{2n+1, 2n}(\lambda) = \pi - 2(2n+1) \lambda \qquad (37)$$

give

$$4 \lambda^2 = \gamma, \qquad e(\gamma) = \gamma . \qquad (38)$$

The above expression are exactly the results of LIEB-LINIGER [4].

In the same way by the use of system (16) we can obtain any approximation we wish.

The characteristic feature of the method of moments is the direct determination of the eigenfunction which has the following form

$$2 \pi g(x) = 1 + i \sum_{n=0}^{\infty} I_{2n}(\lambda) \left\{ \frac{1}{(x + i \lambda)^{2n+1}} - \frac{1}{(x - i \lambda)^{2n+1}} \right\} . \qquad (39)$$

For the determination of the exciton spectrum we require, as we shall see in the next paragraph, the eigenfunction of the point $x = 1$.

The eigenfunction (39) for the value $\lambda > 1$ converges absolutely. Study of the case $\lambda = 0$ is more difficult to obtain because as we observe, λ depends on γ and for $\gamma = 0$ the eigenfunction according to LIEB-LINIGER [4], is no longer analytic.

If, therefore, the ground-state energy is known we can by this calculate other physical quantities, namely the chemical potential μ, the potential energy of particle U, as well as the kinetic energy t.

According to Lieb-Liniger there will be

$$\mu = \frac{\partial E_0}{\partial N} = \varrho^2 \left(3\, e(\gamma) - \gamma \frac{\mathrm{d} e(\gamma)}{\mathrm{d} \gamma} \right), \qquad (40)$$

$$U = \frac{C}{N} \cdot \frac{\partial E}{\partial C} = \varrho^2 \gamma \frac{\mathrm{d} e(\gamma)}{\mathrm{d} \gamma}, \qquad (41)$$

$$t = \frac{E_0}{N} - U = \varrho^2 \left(e(\gamma) - \gamma \frac{\mathrm{d} e(\gamma)}{\mathrm{d} \gamma} \right)$$
$$= \mu(\gamma) - 2 \varrho^2 e(\gamma) . \qquad (42)$$

For the zero approximation (29), (30) and for large values of λ we obtain the same results (3), (32) of Lieb-Liniger.

4. The Excitation Spectrum and the Velocity of Sound

The excitation spectrum for a boson gas and the velocity of sound has been studied extensively by LIEB [10]. The sound velocity U_s can be obtained from the excitation spectrum in the following way:

$$U_s = \lim_{p \to 0} \frac{\partial E(p)}{\partial p} \qquad (43)$$

[10] E. LIEB, Phys. Rev. **130**, 1616 [1963].

where $E(p)$ represents the energy of an elementary excitation of momentum p. Also from the macroscopic definition [11] the velocity U_s is given by the relation,

$$U_s = \left\{ \frac{L}{m\varrho} \cdot \frac{\partial^2 E_0}{\partial L^2} \right\}^{1/2} \tag{44}$$

where E_0 represents the ground-state energy.

The calculation of the excitation energy for particles and holes leads as is well known to the integral equations of form (4). In the case of particles the following equations are valid:

$$\begin{aligned} p &= q + \int_{-K}^{K} J(k)\,\mathrm{d}k\,, \\ E_1 &= -\mu + q^2 + 2\int_{-K}^{K} k\,j(k)\,\mathrm{d}k\,, \\ 2\pi\,j(k) &= 2C\int_{-K}^{K} \frac{j(r)\,\mathrm{d}r}{C^2+(k-r)^2} - \pi - \Theta(q-k) \end{aligned} \tag{45}$$

while for the holes:

$$\begin{aligned} p &= -q + \int_{-K}^{K} j(k)\,\mathrm{d}k\,, \\ E_2 &= \mu - q^2 + 2\int_{-K}^{K} j(k)\,\mathrm{d}k\,, \\ 2\pi\,j(k) &= 2C\int_{-K}^{K} \frac{j(r)\,\mathrm{d}r}{C^2+(k-r)^2} + \pi + \Theta(q-k) \end{aligned} \tag{46}$$

where the function is

$$\Theta(q-k) = -2\,\mathrm{arctg}[(q-k)/C]\,. \tag{47}$$

Lieb examined Eqs. (45) in the region $q \approx K$ for which these by changing the variables,

$$k = K x, \quad j(Kx) \to j(x), \quad q = K s, \quad C = K\lambda, \quad s \geqq 1$$

give

$$\begin{aligned} p &= K\left(s + \int_{-1}^{1} j(x)\,\mathrm{d}x\right), \\ E_1 &= -\mu + K^2\left(s^2 + 2\int_{-1}^{1} x\,j(x)\,\mathrm{d}x\right), \\ 2\pi\,j(x) &= 2\lambda\int \frac{j(y)\,\mathrm{d}y}{\lambda^2+(x-y)^2} - \pi + 2\,\mathrm{arctg}\,\frac{s-x}{\lambda}\,. \end{aligned} \tag{48}$$

The case $s \approx 1$ has been studied in detail by Lieb and the solution of Eq. (48) can be expressed by using the function $g(x,\lambda)$ and its derivatives with respect to x and λ. We will now study the integral equation (48) by the moments method for all values of $s \geqq 1$.

The solution of Eq. (48) using (6) will be of the form:

$$2\pi\,j(x) = -\pi + 2\,\mathrm{arctg}\,\frac{s-x}{\lambda} + i\sum_{n=0}^{\infty} I_n \left\{ \frac{1}{(x+i\lambda)^{n+1}} - \frac{1}{(x-i\lambda)^{n+1}} \right\} \tag{49}$$

where

$$I_m = \int_{-1}^{1} x^m\,j(x)\,\mathrm{d}x\,. \tag{50}$$

In view of (49) Eq. (50) gives

$$2\pi I_m = -\pi\,\frac{1+(-1)^m}{m+1} + 2\int_{-1}^{1} x^m\,\mathrm{arctg}\,\frac{s-x}{\lambda}\,\mathrm{d}x + 2\sum_{n=0}^{\infty} I_n B_{n+1,m}(\lambda)\,. \tag{51}$$

The coefficients $B_{n,m}(\lambda)$ are defined by Eq. (11). The integral

$$A_m = \int_{-1}^{1} x^m\,\mathrm{arctg}\,\frac{s-x}{\lambda}\,\mathrm{d}x \tag{52}$$

on integration gives

$$\begin{aligned} A_m = {} & \frac{1}{m+1}\left\{ \mathrm{arctg}\,\frac{s-1}{\lambda} + (-1)^m\,\mathrm{arctg}\,\frac{s+1}{\lambda} \right\} + \frac{i}{2(m+1)} \sum_{l=0}^{m} \frac{1-(-1)^{m-l+1}}{m-l+1} \{(s-i\lambda)^l - (s+i\lambda)^l\} \\ & + \frac{i}{2(m+1)} \left\{ (s-i\lambda)^{m+1} \ln\frac{s-i\lambda-1}{s-i\lambda+1} - (s+i\lambda)^{m+1} \ln\frac{s+i\lambda-1}{s+i\lambda+1} \right\}. \end{aligned} \tag{53}$$

The system (51) because of (53) is

$$\pi I_m = -\frac{\pi}{2}\cdot\frac{1+(-1)^m}{m+1} + A_m + \sum_{0=u}^{\infty} I_n B_{n+1,m}(\lambda) \tag{54}$$

and this can be solved approximately by the same method as the system (16).

For the calculation of the momentum p and of the excitation energy E_1 the moments I_0 and I_1 are required.

[11] F. LONDON, Superfluids, Vol. II, John Wiley & Sons, New York 1954.

The zero approximation gives the following results:

$$I_0 = \frac{\left\{(1+s)\,\mathrm{arctg}\,\frac{1+s}{\lambda} - (1-s)\,\mathrm{arctg}\,\frac{1-s}{\lambda}\right\} + \frac{\lambda}{2}\ln\frac{\lambda^2+(1-s)^2}{\lambda^2+(1+s)^2} - \pi}{\pi - 2\,\mathrm{arctg}\,\frac{1}{\lambda}}, \tag{55}$$

$$I_1 = \frac{A_1}{\pi} = \frac{1}{2\pi}\left[\mathrm{arctg}\,\frac{s-1}{\lambda} - \mathrm{arctg}\,\frac{s+1}{\lambda} + 2\lambda + \lambda s \ln\frac{\lambda^2+(1-s)^2}{\lambda^2+(1+s)^2} + (s^2-\lambda^2)\left\{\mathrm{arctg}\,\frac{1-s}{\lambda} + \mathrm{arctg}\,\frac{1+s}{\lambda}\right\}\right]. \tag{56}$$

The above expressions for large value of λ become:

$$I_0 = -(\lambda\pi - 2s)/(\lambda\pi - 2), \qquad I_1 = -2/3\,\pi\lambda. \tag{57}$$

From formula (43) we get

$$p = K[s - (\lambda\pi - 2s)/(\lambda\pi - 2)] = K\lambda\pi\cdot(s-1)/(\lambda\pi - 2), \tag{58}$$

$$E_1 = -\mu + K^2(s^2 - 4/3\,\pi\lambda). \tag{59}$$

The calculation of the sound velocity V_s can also be expressed according to LIEB [10] as

$$V_s = \frac{2\varrho}{\lambda}\left\{\frac{1}{\gamma} - \int_{-1}^{1} dx\cdot\frac{dg(x,\lambda)}{d\lambda}\right\} = 2\frac{K}{\gamma}\left\{\frac{1}{\gamma} - \int_{-1}^{1} dx\,\frac{dg(x,\lambda)}{d\lambda}\right\} \tag{60}$$

from which we derive the results of BOGOLIUBOV [12] for small values of λ. In the same way we can examine the energy spectrum of holes.

By the moments method we will also examine the HULTHEN [5] integral equation.

5. Hulthen's Equation

Hulthen's equation is closely related to the theory of antiferromagnetism of a infinite atomic chain. It is known that the problem of ferromagnetism and antiferromagnetism is connected with the spectrum of the following Hamiltonian,

$$H = -\tfrac{1}{2}\{\sigma_x\sigma_x' + \sigma_y\sigma_y' + \Delta\sigma_z\sigma_z'\} \tag{61}$$

where σ and σ' represent the Pauli matrices of an atom and of its neighbouring atom. The parameter Δ represents the anisotropy and for $\Delta = -1$ we have the case of isotropic antiferromagnetism.

[12] C. DE WITT, The Many Body Problem, John Wiley & Sons, New York 1958.

The magnetization Y of each atom is characteristic of the eigenvalue of the operator.

$$\overline{Y} = (1/N)\sum\sigma_z \tag{62}$$

where N is the number of atoms.

The magnetization Y and the ground-state of energy E_0 are given by the relations

$$1 - Y = \int_{-\xi_0}^{\xi_0} f(\xi)\,d\xi, \tag{63}$$

$$E_0 = \int_{-\xi_0}^{\xi_0} (1+\xi^2)^{-1} f(\xi)\,d\xi \tag{64}$$

when the function $f(\xi)$ satisfies Hulthen's integral equation

$$f(\xi) = \frac{2}{\pi}\cdot\frac{1}{1+\xi^2} - \frac{2}{\pi}\int_{-\xi_0}^{\xi_0}\frac{f(x)\,dx}{4+(\xi-x)^2}. \tag{65}$$

The case $Y=0$ corresponds to $\xi_0 = \rightarrow\infty$ and the solution of Eq. (65) is given in the close form

$$f(\xi) = \tfrac{1}{2}\sec(\pi/2)\,\xi, \qquad E_0 = \ln 2. \tag{66}$$

The solution of Hulthen's equation (65) for $\xi_0 = \infty$ can be expressed in a series form, i. e.

$$f(\xi) = \frac{2}{\pi}\cdot\frac{1}{1+\xi^2} + \sum_{n=0}^{\infty} A_n\left\{\frac{1}{(2+i\xi)^{n+1}} + \frac{1}{(2-i\xi)^{n+1}}\right\} \tag{67}$$

when the coefficients A_n satisfy the following recursion system

$$A_n = \frac{(-1)^n}{\pi} + \sum_{k=0}^{\infty} A_k(-1)^{n-k}\,2^{n-k}\,\frac{n(n-1)(n-2)\dots(n-k+1)}{k!}.$$

For $n = 0, 1, 2, \dots,$ we obtain

$$\begin{aligned} A_0 &= -1/2\pi, & A_1 &= 0,\\ A_2 &= -A_0 = 1/2\pi, & A_3 &= 0,\\ A_4 &= 5A_0 = -5/2\pi, & A_5 &= 0,\\ A_6 &= -61A_0 = 61/2\pi, & A_7 &= 0,\\ &\dots \end{aligned} \tag{69}$$

From the above it follows that the function $f(\xi)$ is an even function and this is to be expected on account of the symmetric properties of the problem.

The solution (67) and the ground-state energy are given below:

$$f(\xi) = \frac{2}{\pi} \cdot \frac{1}{1+\xi^2} + \sum_{n=0}^{\infty} A_{2n} \left\{ \frac{1}{(2+i\,\xi)^{2n+1}} + \frac{1}{(2-i\,\xi)^{2n+1}} \right\}, \tag{70}$$

$$E_0 = 1 + 2\,\pi \sum_{n=0}^{\infty} A_{2n}/3^{2n+1} = 1 - \tfrac{1}{3} + \tfrac{1}{27} - \tfrac{5}{243} + \dots = \ln 2\,. \tag{71}$$

Hulthen's equation (65) and relations (64) using the transformation

$$x \to \xi_0\, x\,, \quad \xi \to \xi_0\, y\,, \quad \xi_0 = 2/\lambda\,, \quad f(x\,\xi_0) = F(x) \tag{72}$$

become

$$2\,\pi\, F(y) = \frac{\lambda^2}{y^2 + (\lambda/2)^2} - 2\,\lambda \int_{-1}^{1} \frac{F(x)\,\mathrm{d}x}{\lambda^2 + (x-y)^2}\,, \tag{73}$$

$$\tfrac{1}{2}\lambda(1-Y) = \int_{-1}^{1} F(x)\,\mathrm{d}x\,, \tag{74}$$

$$E_0 = \tfrac{1}{2}\,\lambda \int_{-1}^{1} \frac{F(x)\,\mathrm{d}x}{x^2 + (\lambda/2)^2}\,. \tag{75}$$

Making now use of the moments method we find the solution of Eqs. (73) which is

$$2\,\pi\, F(y) = \lambda^2/[y^2 + (\lambda/2)^2] - i \sum_{n=0}^{\infty} I_{2n} \left\{ \frac{1}{(y+i\,\lambda)^{2n+1}} - \frac{1}{(y-i\,\lambda)^{2n+1}} \right\}. \tag{76}$$

In the present case the coefficients I_{2n} satisfy the system

$$\pi\, I_{2m} = \lambda^2\, L_{2m} - \sum_{n=0}^{\infty} B_{2n+1,\,2m}(\lambda)\; I_{2n}\,. \tag{77}$$

The coefficients $B_{2n+1,\,2m}(\lambda)$ are defined by the relation (11) and

$$L_{2m} = \int_0^1 \frac{x^{2m}\,\mathrm{d}x}{x^2 + (\lambda/2)^2} = \sum_{l=0}^{m-1} \frac{(-\lambda^2/4)^l}{2\,m - 2\,l - 1} + \frac{(-\lambda^2/4)^m}{\lambda/2} \operatorname{arctg}(2/\lambda)\,. \tag{78}$$

On account of the solution (76) conditions (74) and (75) become

$$\tfrac{1}{2}\,\lambda(1-Y) = I_0\,, \tag{79}$$

$$E_0 = \frac{\lambda^3}{4\,\pi} \int_{-1}^{1} \frac{\mathrm{d}x}{(x^2 + \frac{1}{2}\,\lambda^2)^2} - i\,\frac{\lambda}{4\,\pi} \sum_{n=0}^{\infty} I_{2n} \int_{-1}^{1} \frac{x^2 + (\lambda/4)^2}{\mathrm{d}x} \left\{ \frac{1}{(x+i\,\lambda)^{2n+1}} - \frac{1}{(x-i\,\lambda)^{2n+1}} \right\}. \tag{80}$$

The zero ordere approximation of (77) gives

$$I_0 = \frac{2\,\lambda \operatorname{arctg}(2/\lambda)}{\pi + 2 \operatorname{arctg}(1/\lambda)} = \tfrac{1}{2}\,\lambda(1-Y) \tag{81}$$

or,

$$Y = 1 - \frac{4 \operatorname{arctg}(2/\lambda)}{\pi + 2 \operatorname{arctg}(1/\lambda)}\,. \tag{82}$$

The above approximation may be used without restriction for all values of λ.

The case $\lambda = 0$ that is $\xi_0 = \infty$ gives $Y = 0$ which is the case of Hulthen.

Also for large values of $\lambda \gg 1$ the relation (82) gives $Y = 1$.

The ground-state energy for the zero approximation is given by the following formula:

$$2\,\pi\, E_0 = 2\,\lambda \left\{ \frac{1}{1+\lambda^2/4} + (2/\lambda) \operatorname{arctg}(2/\lambda) - \tfrac{4}{3}\left\{2 \operatorname{arctg}(2/\lambda) - \operatorname{arctg}(1/\lambda)\right\} \cdot \frac{4 \operatorname{arctg}(2/\lambda)}{\pi + 2 \operatorname{arctg}(1/\lambda)}\right. . \tag{83}$$

Since the function $4 \operatorname{arctg}(2/\lambda)/[\pi + 2 \operatorname{arctg}(1/\lambda)]$ is equal to 1 for very small values of λ the relation (83) can be written in the present case as

$$2\,\pi\, E_0 = \tfrac{4}{3}\left\{\operatorname{arctg}(2/\lambda) + \operatorname{arctg}(1/\lambda)\right\} + \frac{2}{\lambda} \cdot \frac{1}{1 + \frac{1}{4}\,\lambda^2}\,. \tag{84}$$

The case $\lambda = 0$ gives

$$E_0 = 1 - \tfrac{1}{3}\,, \tag{85}$$

thus giving the first terms of series (71).

More detailed investigation of Hulthen equation is recently given in a new paper [14] in which the spectrum of Hamiltonian (61) is studied for all values of parameter Δ.

[13] C. N. Yang and C. P. Yang, Phys. Rev. **150,** 321, 327 [1966].

[14] A. D. Jannussis, Z. Naturforsch. **24 a,** 762 [1969].

Conclusions

Using the method of moments, which was applied for the solution of the integral equations (3) and (48), the ground-state energy and the excitation spectrum of a one dimensional N particle system, interacting in a $\delta(x_i - x_j)$ potential, have been calculated.

The same method has been applied for the solution of Hulthen's equation (65) from which the ground-state energy and the magnetization of the system have been calculated for the Hamiltonian (61) and for $\Delta = -1$.

The expressions (23) and (30) corresponding to the zero approximation of the moments method can be used for all values of parameter γ and λ except for the case $\gamma = 0$, which must be studied separately.

The approximate results (55) and (56) are valid for all values of the parameter $S \geqq 1$. For large and small values of λ and for $S \approx 1$ we arrive at the well-known results of Lieb.

The most important result is the determination of magnetization expression (82). This expression corresponds to the zero approximation but is valid for all values of λ and in addition gives the actual magnetization curve.

If the above magnitudes are calculated by the use of higher order approximations the real solution of the problems are obtained. This can easily be done since the system of coefficients is linear.

Transmission of a Lorentzian Spectral Line Through a Layer of Lorentzian Absorbers. Part II

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The formalism presented in Part I has been developed further. By analytical methods we have derived a formula for the linewidth

$$W = (\Gamma_S + \Gamma_A)\{1 + \gamma\, T[1 + \gamma\, T/2! - (\gamma+1)^2\, T^2/3! - (7\gamma^3 + 6\gamma^2 - 2)\, T^3/4! + \tfrac{1}{3}(31\gamma^4 + 120\gamma^3 + 156\gamma^2 + 72\gamma + 6)\, T^4/5! + \ldots]\}$$

where $\gamma = \Gamma_A/\Gamma_S$ and T is a new dimensionless parameter which is proportional to the absorber thickness. The application to Mössbauer spectroscopy is discussed. The results are valid for environmental broadening of the Lorentzian type.

8. Summary of Results Obtained in Part I

In Part I of this investigation [15], we considered the total intensity – of an originally Lorentzian line – that is transmitted through a layer of Lorentzian absorbers. This transmitted intensity is given by

$$P(\Delta E) = P(\infty)\ \mathrm{tran}(\gamma, s; x)$$

where the *transmission function* is defined by

$$\mathrm{tran}(\gamma, s; x) = \frac{1}{\pi} \int \frac{\exp\{-s/[1 + (z/\gamma)^2]\}}{1 + (z+x)^2}\, \mathrm{d}z\,.$$

For the definitions of the various symbols, the reader is referred to Sections 1 and 2. We continue to employ the convention, adopted in Section 2, that the limits of an integral are $-\infty$ and $+\infty$ when they are not indicated explicitly.

Our main result was that the transmission function can also be represented by a series,

$$\mathrm{tran}(\gamma, s; x) = \sum_{m=0}^{\nu-1} \frac{(-s)^m}{m!} Q_m(\gamma, x) + \frac{1}{2} \frac{(-s)^\nu}{\nu!} Q_\nu(\gamma, x) \pm \frac{1}{2} \frac{s^\nu}{\nu!} Q_\nu(\gamma, x) \quad \text{for} \quad \nu \geqq s\,. \tag{8.1}$$

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[15] Part I appeared in Z. Naturforsch. **23 a**, 1439 [1968]. Sections, equations and references of Part II are numbered consecutively after those of Part I.